Improvements to Static Probabilistic Timing Analysis for Systems with Random Cache Replacement Policies

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Static Probabilistic Timing Analysis (SPTA)

- Aim is to show that the probability of timing failure falls below some threshold e.g. $10^{-9}$ failures per hour: $p_{WCET}$ v. budget

![Diagram of CPU, Program, Instruction Cache, Memory, and Random replacement policy.}

Probabilistic WCET ($p_{WCET}$) distribution

![Graph showing probability of execution time.]
pWCET distribution (1-CDF)

- pWCET without pre-emption
- pWCET with 1 pre-emption
- Upper bound ≤ relationship
Simple model of execution

- Instructions are either:
  - Cache hit or cache miss
  - Misses take longer ($H = 1$ cycle, $M = 10$ cycles)

- Fully associative cache of $N$ blocks
  - Memory blocks can be loaded into any block in cache
  - Each instruction resides in a memory block

- On a cache miss
  - Random choice of cache block to evict
  - Evict that block, load the requested block into the evicted location

- Probability of a cache hit:
  \[ P_{hit}(k) = \left( \frac{N - 1}{N} \right)^k \]
  (when $k < N$ otherwise 0)

  - $k$ is **re-use distance** = number of intervening evictions since the memory block was last loaded into cache
Static Probabilistic Timing Analysis (for single path programs)

- Sequence of instructions represented by their memory blocks \( a, b, a^1, c, d, b^3, c^2, d^2, a^5 \)

- Get a probability distribution (pWCET) for each instruction
  - Depends only on re-use distance \( k \)
  - Possible to model instructions as independent, hence we can convolve distributions for instructions to get a pWCET distribution for a sequence of instructions

E.g. two instructions with \( P_{hit} = 0.8 \) and 0.7

\[
\begin{pmatrix}
1 & 10 \\
0.8 & 0.2
\end{pmatrix}
\otimes
\begin{pmatrix}
1 & 10 \\
0.7 & 0.3
\end{pmatrix}
= \begin{pmatrix}
2 & 11 & 20 \\
0.56 & 0.38 & 0.06
\end{pmatrix}
\]
pWCET distribution (1-CDF)
SPTA has some pessimism

- Sequence of instructions represented by their memory blocks: $a, b, a^1, c, d, b^3, c^2, d^2, a^5$

- Consider the $a^5$
  - 5 because of the intervening instructions $c, d, b^3, c^2, d^2$
  - $c, d,$ are definitely misses
  - $b^3, c^2, d^2$ considered as misses when analysing $a^5$

**Pessimistic** because the probability that $b^3, c^2, d^2$ are all misses is already $< 7.1 \times 10^{-7}$ (with $N = 256$)

*How can we obtain a tighter pWCET that is still correct (not optimistic)?
Reducing the pessimism


\[ P_{hit} = \left( \frac{N-1}{N} \right) \sum P_{\text{miss}} \]

- But is it correct?
  
  Consider \( a, b, a^1, b^1 \) with \( N = 2 \)

  \[ P_{hit} = \left( \frac{1}{2} \right)^{1/2} = 1/\sqrt{2} \]

  **Irrational value for a probability?**
Counter example: Analysis from [5]

Consider $a, b, a^1, b^1$ with $N=4$

- Distributions for $a, b, a^1$
  
  $\begin{pmatrix} 10 \\ 1 \end{pmatrix} \begin{pmatrix} 10 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 10 \\ 0.75 & 0.25 \end{pmatrix}$

- For $b^1$ $\left(\frac{3}{4}\right)^{0.25}$ = 0.9306 according to [5]

- Hence
  
  $\begin{pmatrix} 10 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 10 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 10 \\ 0.75 & 0.25 \end{pmatrix} \otimes \begin{pmatrix} 0.9306 & 0.0694 \end{pmatrix} = \begin{pmatrix} 22 & 31 & 40 \\ 0.69795 & 0.2847 & 0.01735 \end{pmatrix}$
Consider $a, b, a_1, b_1$ with $N = 4$. Two cases:

- **Case 0:** $a_1$ is a hit (probability of occurrence = 0.75)
  - Given that $a_1$ is a hit then $b_1$ is guaranteed to also be a hit
  
  \[
  \text{Partial pWCET} = \begin{pmatrix}
  10 \\
  1
  \end{pmatrix} \times \begin{pmatrix}
  10 \\
  1
  \end{pmatrix} \times \begin{pmatrix}
  1 \\
  0.75
  \end{pmatrix} \times \begin{pmatrix}
  1 \\
  1
  \end{pmatrix} = \begin{pmatrix}
  22 \\
  0.75
  \end{pmatrix}
  \]

- **Case 1:** $a_1$ is a miss (probability of occurrence = 0.25)
  - Given that $a_1$ is a miss then $b_1$ has $P_{hit} = 0.75$

  \[
  \text{Partial pWCET} = \begin{pmatrix}
  10 \\
  1
  \end{pmatrix} \times \begin{pmatrix}
  10 \\
  1
  \end{pmatrix} \times \begin{pmatrix}
  10 \\
  0.25
  \end{pmatrix} \times \begin{pmatrix}
  1 \\
  0.75
  \end{pmatrix} \times \begin{pmatrix}
  10 \\
  0.25
  \end{pmatrix} = \begin{pmatrix}
  31 \\
  0.1875
  \end{pmatrix}
  \]

- **Overall pWCET**

  \[
  \begin{pmatrix}
  22 & 31 & 40 \\
  0.75 & 0.1875 & 0.0625
  \end{pmatrix}
  \]

This is precise – we covered all possibilities
Consider \( a, b, a^1, b^1 \) with \( N = 4 \).

- Precise analysis:
  
  \[
  \begin{pmatrix}
  22 & 31 & 40 \\
  0.75 & 0.1875 & 0.0625 \\
  \end{pmatrix}
  \]

  Exact but exponential complexity

- Analysis from [5]:
  
  \[
  \begin{pmatrix}
  22 & 31 & 40 \\
  0.69795 & 0.2847 & 0.01735 \\
  \end{pmatrix}
  \]

  OPTIMISTIC

- Simple analysis from [3]:
  
  \[
  \begin{pmatrix}
  22 & 31 & 40 \\
  0.5625 & 0.375 & 0.0625 \\
  \end{pmatrix}
  \]

  Pessimistic but Ok
Open Problem: Can we tighten the pWCET (1-CDF) found by SPTA?

- Sequence of instructions represented by their memory blocks $a, b, a^1, c, d, b^3, c^2, d^2, a^5$ with re-use distances
- Probability of a hit for a single instruction (for $k < N$)

$$P_{hit}(k) = \left(\frac{N-1}{N}\right)^k$$

- Convolve pWCET distributions for individual instructions to get overall pWCET distribution for the sequence
- Existing analysis is simple but somewhat pessimistic as intervening instructions are not necessarily certain to be misses
- Precise analysis is exponential in complexity

Can we find a tighter upper bound on the pWCET that can be computed efficiently?
Extent of the pessimism

Can we close this gap?

Pessimism

Execution time

Probability

- PROG
- P*
- nxP*