Of Walks and Graphs

An Introduction to Walk Theory

October 2015
Outline

1 Introduction
   - Why walks?
   - Strange observations

2 Walk Theory
   - Prime factorisation of walks
   - Posets of walks

3 Algebraic Walk Theory
   - Functions on walk posets
   - Path-sums

4 Primes for Applications
   - Dynamics of wealth exchange
   - Polymers & percolation

5 Conclusion
Why Walks?

Walk: a *trajectory* on a graph
Not necessarily random!
Why Walks?

Walk: a *trajectory* on a graph
Not necessarily random!

Why walks?

Network analysis is often walk-based
Processes undergone by physical systems

Why Walks?

Walks are pervasive objects!

- Arbitrary matrix $\mathcal{M}^n = \text{sum of walk weights}$

$$\mathcal{M} = \begin{pmatrix} 1 & i \\ -3 & 5 \end{pmatrix}$$

$$\left( \sum_{n=1}^{3} \mathcal{M}^n \right) = \sum_{n=1}^{3} \mathcal{M}^n$$

$\rightarrow$ Matrix power series are walk-series

Analytic matrix function $f(\mathcal{M}) = \text{series of walk weights}$
Strange Observation 1

- Changing a square lattice

\[ \# W_{\bullet \to \bullet'}(\ell) \longrightarrow \# W_{\bullet \to \bullet'}(2\ell) \]
Strange Observation 1

- Changing a square lattice

\[ \# W_{\bullet \rightarrow (\ell)} \longrightarrow \# W_{\bullet \rightarrow (2\ell)} \]

\[ \text{Non-trivial for graphs: regularity, non-directedness lost} \]
\[ \text{Trivial transformation for walks} \]
Strange Observation 2

Which graphs are “similar”?
Strange Observation 2

Which graphs are “similar”? 

Walk sets on these graphs are \textit{identical} (I will come back to this)
Strange Observations

- Completely *dissimilar* graphs can have the *same* walk sets
- Even *fundamental* graph properties, *regularity*, may have little effect on walks
- Existence of *non-trivial* properties of walks *valid on all* (multi-di)graphs

→ Can we detect similarities between different graphs?

[Diagrams of various graphs]
Strange Observations

- Completely *dissimilar* graphs can have the *same* walk sets
- Even *fundamental* graph properties, *regularity*, may have little effect on walks
- Existence of *non-trivial* properties of walks *valid on all* (multi-di)graphs

→ Can we detect similarities between different graphs?

→ Can we represent and manipulate walks without their graphs?
Elements of Walk Theory

1. Introduction
   - Why walks?
   - Strange observations

2. Walk Theory
   - Prime factorisation of walks
   - Posets of walks

3. Algebraic Walk Theory
   - Functions on walk posets
   - Path-sums

4. Primes for Applications
   - Dynamics of wealth exchange
   - Polymers & percolation

5. Conclusion
Observation: walk = simple path & simple cycle

Walk \( w \) ‘factors’ into 1 simple path and 2 simple cycles

\[ w = \gamma_0 \leftarrow (\gamma_1 \leftarrow \gamma_2) \]

What can we say in general?
Nesting product

- Consider loop-erasing

Inverse of loop-erasing: nesting, product \( \odot \)

\[
\begin{align*}
\alpha & \quad (w_1 \odot w_2) = (w_1 \odot w_2) \\
\text{or} & \\
\alpha & \quad (w_1 \odot w_2) = (w_1 \odot w_2)
\end{align*}
\]

\[w = \gamma_0 \odot (\gamma_1 \odot \gamma_2)\]

Does not visit any of these vertices except \( \alpha \)
The Fundamental Theorem

**Theorem**

Let $G$ be a graph and $w$ a walk on $G$. Then there exists a **unique factorisation** of $w$ into $\odot$-products of **prime walks**, the simple cycles on $G$.

**Prime walks:** \( \gamma \mid w \odot w' \Rightarrow \gamma \mid w \) or \( \gamma \mid w' \)

\[ 1232332121 = (121 \odot (232 \odot (232 \odot 33))) \odot 121 \]

\[ 1 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow \circ \circ \text{ complicated: non-commutative, non associative} \]
Any trajectory on $G$

Let edges $e_{ij}$, and $e_{kl}$ commute if $i \neq k$

Resulting set: \textbf{Words} (connected words = walks)
The Fundamental Theorem 2

Any trajectory on $\mathcal{G}$

Let edges $e_{ij}$, and $e_{kl}$ commute if $i \neq k$

Resulting set: Words (connected words = walks)

Theorem

Let $\mathcal{G}$ be a graph and $c$ a word on $\mathcal{G}$. Then there exists a unique factorisation of $c$ into $\cdot$-products of prime walks, the simple cycles on $\mathcal{G}$.

Prime walks: $\gamma | c \cdot c' \Rightarrow \gamma | c$ or $\gamma | c'$
Posets of Walks

Construct a prime-based representation of walks

- Sets of words ordered by divisibility

\[ c \mid c' \iff c \leq c' \]

- Example: poset of words \( P_G = (\mathcal{W}_G, \leq) \) on \( G = \)

\[ \begin{array}{c}
3 \\
2 \\
2
\end{array} \]

How does \( P_G \) relate to \( G \)?
Posets of Walks

Construct a prime-based representation of walks

- Sets of words ordered by divisibility
  \[ c | c' \iff c \leq c' \]

- Example: poset of words \( P_G = (\mathcal{W}_G, \leq) \) on \( G = \)

How does \( P_G \) relates to \( G \)?
Theorem
For any graph $G$, there exists infinitely many non-isomorphic graphs $G_1, G_2, \ldots$ whose sets of walks are isomorphic

$$(W_G, \cdot) \simeq (W_{G_1}, \cdot) \simeq (W_{G_2}, \cdot) \simeq \cdots$$

In particular $P_G = P_{G_1} = P_{G_2} = \cdots$
Isomorphic Walk Sets

Isomorphic walk sets on dissimilar graphs

Theorem (courtesy of ⊙)

For any graph $G$ and vertices $\alpha, \beta \in V(G)$, there exists infinitely many non-isomorphic graphs $G_i$ and vertices $a_i, b_i \in V(G_i)$ with

$$(W_G; \alpha\beta, \circ) \cong (W_{G_i}; a_ib_i, \circ)$$
Isomorphic Walk Sets

Invariance of $P_G$ under graph transformations

Corollary

Let $G$ be a graph and $G'$ the graph $G$ with some added vertices on edges. Then $P_G = P_{G'}$, 

$$(W_G, \cdot) \simeq (W_{G'}, \cdot)$$
Isomorphic Walk Sets

Invariance of $P_G$ under graph transformations

Corollary

Let $G$ be a graph and $G'$ the graph $G$ with some added vertices on edges. Then $P_G = P_{G'}$, $(W_G, \cdot) \simeq (W_{G'}, \cdot)$

More $P_G$-invariant transformations exist! ‘Broad’ classification possible
Isomorphic Walk Sets

$P_G$-invariant graph transformations as *generators*

$P_G$ a *graph-free* representation of walk sets?

$\rightarrow$ Depends weakly on any specific $G$

$\rightarrow$ Walks without graphs?
Strange Observation 3

- Number of walk posets on $n$ primes:
  
  \[ B_n \sim \left( \frac{0.792 \cdot n}{\log n} \right)^n \]

\( \subseteq P_G; \)
Strange Observation 3

- Number of walk posets on $n$ primes:
  \[ B_n \sim \left( \frac{.792}{\log n} n \right)^n \]

\[ \subseteq P_{G; \bullet} \]

No graph exists with exactly this walk poset

\( a, b, c, d \) primes

\( \emptyset \)

\( \{a\} \)

\( \{a, b\} \)

\( \{a, b, c\} \)

\( \{a, b, c, d\} \)

\( \{a, d\} \)

\( \{a, b, d\} \)

\( \{a, b, c\} \)

\( \{b\} \)

\( \{b, c\} \)

\( \{b, c, d\} \)
Strange Observation 3

- Number of walk posets on $n$ primes:
  \[ B_n \sim (0.792 \, n / \log n)^n \]

- Conjectured # posets on $n$ primes with exact graph realization
  \[ C_n \sim 4^n n^{-3/2} / \sqrt{\pi} \ll B_n \]

\[ \Rightarrow \text{Most walk sets have no exact graph realization!} \]
1 Introduction
   ▪ Why walks?
   ▪ Strange observations

2 Walk Theory
   ▪ Prime factorisation of walks
   ▪ Posets of walks

3 Algebraic Walk Theory
   ▪ Functions on walk posets
   ▪ Path-sums

4 Primes for Applications
   ▪ Dynamics of wealth exchange
   ▪ Polymers & percolation

5 Conclusion
Goal: use functions to represent $P_G$

Easier to manipulate & compare

Functions on $P_G$ (matrices)

$$w \not\geq w' \implies F(w, w') = 0$$

Simplest description of $P_G$

$$Z(w, w') = 1 \iff w \leq w'$$

Preserves the division!

$$w' \cdot w = w_2 \cdot w_1 \iff F(w, w') = F(w_1, w_2)$$

$\Rightarrow R P_G = $ all such functions, equipped with matrix product
**Functions on walk poset**

**Goal:** use functions to represent $P_G$

$\mapsto$ Easier to manipulate & compare

Functions on $P_G$ (matrices)

$$w \not\preceq w' \implies F(w, w') = 0$$

Simplest description of $P_G$

$$Z(w, w') = 1 \iff w \preceq w'$$

Preserves the division!

$$\frac{w'}{w} = \frac{w_2}{w_1} \iff F(w, w') = F(w_1, w_2)$$

- $RP_G = \text{all such functions, equipped with matrix product}$
- $Z \in RP_G$
Theorem (Giscard, Rochet, Espinasse)

Let $G$ be a graph. Then $R P_G$ is isomorphic to the algebra of formal series on words on $G$. The representation of $Z$ is

$$Z \longrightarrow \zeta = \sum_c c,$$

and

$$\zeta = \frac{1}{\det(I - W)}.$$

Furthermore, $\zeta$ determines $G$ uniquely, up to isomorphism.
Walk Zeta function

**Theorem** (Giscard, Rochet, Espinasse)

Let $G$ be a graph. Then $RP_G$ is **isomorphic** to the **algebra of formal series** on words on $G$. The representation of $Z$ is

\[ Z \rightarrow \zeta = \sum_c c, \]

and

\[ \zeta = \frac{1}{\det(I - W)}. \]

Furthermore, $\zeta$ determines $G$ **uniquely**, up to isomorphism.

Relations between functions on $RP_G$ yield the **theory of walks**
“Relations between functions on $RP_G$ yield the theory of walks”

Example: divisor counting

- Let $w$ be a walk, $d(w)$ number of its divisors

$$d(w) = \sum_{w'|w} 1 = \sum_{w'} Z(1, w') Z(w', w) = Z^2(1, w)$$
Algebraic Walk Theory

“Relations between functions on $RP_G$ yield the theory of walks”

Example: divisor counting

- Let $w$ be a walk, $d(w)$ number of its divisors

$$d(w) = \sum_{w' | w} 1 = \sum_{w'} Z(1, w')Z(w', w) = Z^2(1, w)$$

- $d(w) = \zeta^2[w]$  

Relation valid on all (hyper)(continuous)(multi-di)graphs...
Corollary

Let $G$ be the graph $G_\mathbb{N}$. Then $RP_G$ is the algebra of Dirichlet series. In particular

$$\zeta = \zeta_R(s) = \sum_n \frac{1}{n^s} \quad \text{and} \quad Z^{-1} \rightarrow \zeta^{-1} = \sum_n \frac{\mu(n)}{n^s}$$

Number theory from walks!

- Divisor counting
  $$\zeta^2 \rightarrow \zeta_R(s)^2$$

- Connected words (walks)
  $$\log \zeta \rightarrow \sum_n \frac{\Lambda(n)}{\log n} \frac{1}{n^s}$$

- Square-free
  $$|\zeta^{-1}| \rightarrow \zeta_R(s)/\zeta_R(2s)$$
Algebraic walk theory

Looking for patterns in a graph?

\[
\frac{(\zeta - 1)^{q+1}}{\zeta_q} _2F_1(1, q + 1; q + 2; 1 - \zeta)
\]

\[q = 0\] eliminates non-connected words

\[q = 1\] eliminates flowers

\[q = 2\] eliminates cycles of cycles...

\[\zeta\]-series answer length-independent questions
Functions on graphs: path-sums

- Network analysis idea:
  \( G \) a graph, put weights \( G \to G \), calculate \( F(G) \)

**Theorem (Courtesy of ⊙)**

*Let \( F \) be an analytic function and \( G \) a weighted graph. Then \( F(G) \) has a representation involving only prime walks, which depends only on \( P_G \) and not on \( F \).*
Functions on graphs: path-sums

- Network analysis idea:
  \( G \) a graph, put weights \( G \to G \), calculate \( F(G) \)

**Theorem (Courtesy of ⊙)**

*Let \( F \) be an analytic function and \( G \) a weighted graph. Then \( F(G) \) has a representation involving only prime walks, which depends only on \( P_G \) and not on \( F \).*

- Consequence: manipulate \( F(G) \) at will

Red vertices have the same resolvent & exponential centrality...
All self-communicability measures are the same!
Functions on graphs: path-sums

- Network analysis idea 2:
  
  dynamical weights!

**Theorem**

Let $G(t)$ be a dynamically weighted graph. Then the solution $S(t)$ of

$$\frac{d}{dt} S(t) = S(t) G(t)$$

has a representation involving only prime walks, which depends only on $P_G$.

→ Probably true for more complex differential equations
Primes for Applications

1. Introduction
   - Why walks?
   - Strange observations

2. Walk Theory
   - Prime factorisation of walks
   - Posets of walks

3. Algebraic Walk Theory
   - Functions on walk posets
   - Path-sums

4. Primes for Applications
   - Dynamics of wealth exchange
   - Polymers & percolation

5. Conclusion
Dynamics of wealth exchange

Source: Media Industry Networks 2006, Laurie Lock Lee, Optimice
Dynamics of wealth exchange

"Distance" between poset $P_G$:

Clusters vertices by dynamical properties

→ Clusters vertices by dynamical properties
Dynamics of wealth exchange

"Distance" between poset \( P_G \):

Clusters vertices by dynamical properties

- Verizon more central than Time Warner
- Time Warner shows up more often in \( P_{\text{Google, Yahoo, Microsoft}} \) than Verizon
Dynamics of wealth exchange

Return to equilibrium: let’s perturb Verizon and Time Warner

Time Warner has more impact than Verizon!
Polymers & percolation

- Square-lattice:
  
  *How many self avoiding walks of length $\ell$?*

- Even asymptotic results are beyond reach since 1953

- Numerical answer for planar lattices: $\mu^\ell \ell^{11/32}$
Polymers & percolation

▶ Square-lattice:

*How many self avoiding walks of length $\ell$?*

▶ Even asymptotic results are beyond reach since 1953

▶ Numerical answer for planar lattices: $\mu^\ell \ell^{11/32}$

Prime Walk Theorem:

▶ How many primes of length $\ell$ are there?

▶ $P_g$ is the same on lattices
Conclusion

1. Introduction
   - Why walks?
   - Strange observations

2. Walk Theory
   - Prime factorisation of walks
   - Posets of walks

3. Algebraic Walk Theory
   - Functions on walk posets
   - Path-sums

4. Primes for Applications
   - Dynamics of wealth exchange
   - Polymers & percolation

5. Conclusion
Conclusion

Main Message

Walks are not slaves to graphs and need to be treated by a separate approach from graph theory; a theory of walks.

Results

- Prime factorization of walks
- Graph-free approach to walks using prime orderings
- HUGE reservoir of algebraic machinery

Open problems

- Understanding exact graph realizations
- Classify $P_G$-invariant transformations
Thank You!

Collaborators

S. Thwaite  Z. Choo  P. Rochet  T. Espinasse
Ihara Zeta Function?

Primitive orbits **ARE NOT PRIME** $p | a \cdot b \nRightarrow p | a$ or $p | b$

$\iff$ Factorization is not unique

$$\zeta_I = \prod_{\ell} (1 - u^\ell)^{-\pi(\ell)}$$

$\pi(\ell) = \text{number of primitive orbits, length } \ell$

The Dedekind zeta function for perfect walk powers is the Ihara zeta $\zeta_I$. In particular

$$\pi(\ell) = \frac{1}{\ell} \sum_{n \mid \ell} \mu(\ell/n) \text{Tr}[A^n]$$

We have found that the Ihara zeta function possesses many analogous properties to the Dedekind zeta function of an algebraic number field.