Walks are just connected hikes

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OUTLINE

WALKS ON A GRAPH AND LINEAR ALGEBRA

HIKES INCIDENCE ALGEBRA

SPECTRAL ANALYSIS

OPEN HIKES
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WALKS ON A GRAPH AND LINEAR ALGEBRA

HIKES INCIDENCE ALGEBRA

SPECTRAL ANALYSIS

OPEN HIKES
• $G = (V, E)$ be a directed graph

• $A(x)$ its variable adjacency matrix.

$$A(x) = \begin{bmatrix}
0 & x_{12} & x_{13} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_{23} & 0 & x_{25} & 0 & 0 & 0 \\
x_{31} & 0 & 0 & x_{34} & 0 & 0 & 0 & 0 \\
x_{41} & 0 & 0 & x_{44} & 0 & 0 & 0 & 0 \\
0 & x_{52} & 0 & 0 & 0 & 0 & x_{57} & 0 \\
0 & 0 & 0 & x_{65} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_{76} & 0 & 0 & 0
\end{bmatrix}$$
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DEFINITIONS

1. A walk is a sequence of contiguous edges
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\[ W = x_{34} \]
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2. A cycle is a walk that ends at its starting point (multisets of left and right indices are equal)

3. A hike is a sequence of edges such that the multisets of left and right indices are equal

\[ W = x_{34}x_{44} \]
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$$h = x_{13}x_{34}x_{41}x_{25}x_{52}$$
A hike $h$ is:

- *connected* if its edges can be reordered contiguously
- *self-avoiding* if it does not cross the same vertex twice
- *prime* if it is connected and self-avoiding
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**Proposition (Prime Decomposition)**

Every hike can be reordered into a sequence of simple cycles *without permuting two edges with the same starting point*. This representation is unique, up to permutations of vertex-disjoint cycles.
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\[
h = (x_{25}x_{52}) (x_{13}x_{34}x_{41}) (x_{12}x_{23}x_{31})
\]

(vertex-disjoint)
Rule: Two edges commute if, and only if, they have different starting points

**Semi-commutative structure of hikes** \((\mathcal{H}, \times)\)

Two hikes \(h_1, h_2\) commute iff they are vertex-disjoint
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**Semi-commutative structure of hikes** \((\mathcal{H}, \times)\)

Two hikes \(h_1, h_2\) commute iff they are vertex-disjoint

\[
x_{25}x_{52} \times x_{13}x_{34}x_{41} = x_{13}x_{34}x_{41} \times x_{25}x_{52}
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**Rule:** Two edges commute if, and only if, they have different starting points

**Semi-commutative structure of hikes** \((\mathcal{H}, \times)\)

Two hikes \(h_1, h_2\) commute iff they are vertex-disjoint

\[
x_{25} x_{52} \times x_{12} x_{23} x_{31} \neq x_{12} x_{23} x_{31} \times x_{25} x_{52}
\]
Let

- $\ell(h)$ the length of $h$ (its degree)
- $\Omega(h)$ the number of primes composing it

**Proposition**

$$\det(-A(x)) = \sum_{h \in \mathcal{H}} \left( -1 \right)^{\Omega(h)} h$$

$h$ self-avoiding $\ell(h)=N$
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**Example**

$$\det(-A(x)) =$$
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- $\ell(h)$ the length of $h$ (its degree)
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**Proposition**

$$\det(-A(x)) = \sum_{h \in \mathcal{H}, \text{ h self-avoiding}} (-1)^{\Omega(h)} h$$

**Example**

$$\det(-A(x)) = (x_{12}x_{23}x_{34}x_{41}) \times (x_{57}x_{76}x_{65})$$
Let
- $\ell(h)$ the length of $h$ (its degree)
- $\Omega(h)$ the number of primes composing it

**Proposition**

$$\det(-A(x)) = \sum_{h \in H \atop h \text{ self-avoiding} \atop \ell(h)=N} (-1)^{\Omega(h)} h$$

**Example**

$$\det(-A(x)) = (x_{12}x_{23}x_{34}x_{41}) \times (x_{57}x_{76}x_{65})$$

$$- (x_{12}x_{23}x_{31}) \times x_{44} \times (x_{57}x_{76}x_{65})$$
**Corollary**

\[
\det(I - zA(x)) = \sum_{\substack{h \in \mathcal{H} \\
 h \text{ self-avoiding}}} (-1)^{\Omega(h)} h z^{\ell(h)}
\]

Define \( \mu : \mathcal{H} \rightarrow \{-1, 0, 1\} \)

\[
\mu(h) = \begin{cases} 
(-1)^{\Omega(h)} & \text{if } h \text{ is self-avoiding} \\
0 & \text{otherwise}
\end{cases}
\]

Looks like a Mobius function...
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SPECTRAL ANALYSIS

OPEN HIKES
**Division in $\mathcal{H}$**

\[ h = d \times h' \iff h' = \frac{h}{d}, \quad h, d, h' \in \mathcal{H}. \]

**Incidence algebra**

Set of functions $f : \mathcal{H} \to \mathbb{R}$, endowed with the Dirichlet convolution

\[ f \ast g(h) = \sum_{d \mid h} f(d)g\left(\frac{h}{d}\right). \]

*Why the Dirichlet convolution?*
DIVISION IN $\mathcal{H}$

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INCIDENCE ALGEBRA
Set of functions $f : \mathcal{H} \to \mathbb{R}$, endowed with the Dirichlet convolution

\[ f \ast g(h) = \sum_{d \mid h} f(d)g\left(\frac{h}{d}\right). \]

Multiplication of formal series

\[
\left( \sum_{h \in \mathcal{H}} f(h)h \, z^{\ell(h)} \right) \times \left( \sum_{h \in \mathcal{H}} g(h)h \, z^{\ell(h)} \right) = \sum_{h \in \mathcal{H}} f \ast g(h) \, h \, z^{\ell(h)}
\]
The Dirichlet convolution is

1. associative
2. distributive over addition
3. not commutative

**IMPORTANT FUNCTIONS**

- The identity: \( \delta(1) = 1 \) and \( \forall h \neq 1, \delta(h) = 0 \)
- The characteristic function: \( \zeta(h) = 1 \)
- The Mobius function: \( \zeta^{-1} \)
**Theorem (Mobius inversion)**

The function

\[ \mu(h) = \begin{cases} (-1)^{\Omega(h)} & \text{if } h \text{ is self-avoiding} \\ 0 & \text{otherwise} \end{cases} \]

is the Mobius function on \((\mathcal{H}, \times)\):

1. \( \forall h \neq 1, \mu \ast \zeta(h) = \sum_{d \mid h} \mu(d) = 0 \)

2. \( \left( \sum_{h \in \mathcal{H}} \mu(h) h z^{\ell(h)} \right) \times \sum_{h \in \mathcal{H}} \zeta(h) h z^{\ell(h)} = \det(I - zA(x)) \times \sum_{h \in \mathcal{H}} h z^{\ell(h)} = 1 \)
ANALOGY WITH \( \mathbb{N} \)

Natural integer \( n = p_1 \ldots p_k \)

- \( \mu(n) = \begin{cases} (-1)^k & \text{if } p_i \neq p_j \\ 0 & \text{otherwise} \end{cases} \)
- \( n_1, n_2 \) co-prime

\[
\mu(n_1)\mu(n_2) = \mu(n_1 n_2)
\]

Hike \( h = c_1 \times \ldots \times c_k \)

- \( \mu(h) = \begin{cases} (-1)^k & \text{if } c_i \cap c_j = \emptyset \\ 0 & \text{otherwise} \end{cases} \)
- \( h_1, h_2 \) disjoint

\[
\mu(h_1)\mu(h_2) = \mu(h_1 \times h_2)
\]

**CO-PRIME = VERTEX-DISJOINT \neq DIFFERENT**

\[ p_1 \neq p_2 \implies p_1, p_2 \text{ co-prime.} \]
Graph representation of $\mathbb{N}$
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- The primes:
Graph representation of $\mathbb{N}$

- The primes: $c_1$
Graph representation of \( \mathbb{N} \)

- The primes: \( c_1, c_2 \)
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- The primes: \( c_1, c_2, c_3 \)
Graph representation of $\mathbb{N}$

- The primes: $c_1$, $c_2$, $c_3$, ...

- Different primes are disjoint: all closed hikes commute
Graph representation of $\mathbb{N}$

- The primes: $c_1, c_2, c_3, ...$

- Different primes are disjoint: all closed hikes commute
**GENERAL CASE**

- The primes: $c_1, c_2, c_3, c_4, c_5, c_6, c_7$

- $c_3$ and $c_6$ are co-prime (disjoint), they commute: $c_3 \times c_5 = c_5 \times c_3$

- $c_3$ and $c_5$ are not co-prime: $c_3 \times c_4 \neq c_4 \times c_3$
GENERAL CASE

• The primes: $c_1$
**General Case**

- The primes: $c_1$, $c_2$
GENERAL CASE

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- The primes: $c_1, c_2, c_3, c_4$
GENERAL CASE

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- $c_3$ and $c_5$ are not co-prime: $c_3 \times c_4 \neq c_4 \times c_3$
WHAT ABOUT CONNECTED HIKES (CYCLES)?

Let $\Lambda(h)$ denote the number of connected representations of $h$. 
What about connected hikes (cycles)?

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Example
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Example

$$c_3 \times c_4 = (x_{54}x_{46}x_{65})(x_{62}x_{26})$$
What about connected hikes (cycles)?

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**Example**

$$c_3 \times c_4 = (x_{54} x_{46} x_{65}) (x_{62} x_{26}) = x_{65} x_{54} x_{46} x_{62} x_{26}$$
What about connected hikes (cycles)?

Let \( \Lambda(h) \) denote the number of connected representations of \( h \)

**Example**

\[
\begin{align*}
\mathbf{c}_3 \times \mathbf{c}_4 &= (x_{54} x_{46} x_{65}) (x_{62} x_{26}) \\
&= x_{65} x_{54} x_{46} x_{62} x_{26} \\
&= x_{26} x_{65} x_{54} x_{46} x_{62}
\end{align*}
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\Lambda(c_3 \times c_4) = 2
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\end{align*}
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\[
\begin{align*}
c_4 \times c_3 &= (x_{62}x_{26})(x_{54}x_{46}x_{65}) \\
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&= x_{26} x_{65} x_{54} x_{46} x_{62} \\
\mathbb{c}_4 \times \mathbb{c}_3 &= (x_{62} x_{26}) (x_{54} x_{46} x_{65}) \\
&= x_{62} x_{65} x_{54} x_{46} \\
&= x_{46} x_{62} x_{26} x_{65} x_{54} \\
&= x_{54} x_{46} x_{62} x_{26} x_{65}
\end{align*}
\]

\[
\Lambda(\mathbb{c}_3 \times \mathbb{c}_4) = 2
\]
What about connected hikes (cycles)?

Let $\Lambda(h)$ denote the number of connected representations of $h$

**Example**

$c_3 \times c_4 = (x_{54} x_{46} x_{65}) (x_{62} x_{26})$

$= x_{65} x_{54} x_{46} x_{62} x_{26}$

$= x_{26} x_{65} x_{54} x_{46} x_{62}$

$\Lambda(c_3 \times c_4) = 2$

$c_4 \times c_3 = (x_{62} x_{26}) (x_{54} x_{46} x_{65})$

$= x_{62} x_{26} x_{65} x_{54} x_{46}$

$= x_{46} x_{62} x_{26} x_{65} x_{54}$

$= x_{54} x_{46} x_{62} x_{26} x_{65}$

$\Lambda(c_4 \times c_3) = 3$
**Theorem**

For all $h \in \mathcal{H}$,

\[
\sum_{d \mid h} \Lambda(d) = \ell(h) \quad \text{and} \quad \sum_{d \mid h} \ell(d) \mu\left(\frac{h}{d}\right) = \Lambda(h)
\]

1. $\Lambda \ast \zeta = \ell \iff \ell \ast \mu = \Lambda$

2. $\Lambda$ is the von Mangoldt function

3. $\ell(h)$ is the “logarithm” of a hike
On $\mathbb{N}$:

1. Connected hikes are powers of primes

2. $\Lambda(h) = \begin{cases} \ell(p) & \text{if } h = p^k \\ 0 & \text{otherwise} \end{cases} \implies$ this is the von Mangoldt function on $\mathbb{N}$!

3. Additive property of $\ell$: $\ell(h \times h') = \ell(h) + \ell(h') \implies \ell \equiv \log$

The von Mangoldt function is a characteristic function over connected hikes.
I) Generalization of number theory: \( \zeta(z) = \det(I - zA(x))^{-1} \)

- \( \zeta^2(z) = \sum_{h \in \mathcal{H}} d(h) h \ z^{\ell(h)} \) with \( d(.) \) the number of divisors
I) Generalization of number theory: $\zeta(z) = \det(I - zA(x))^{-1}$

- $\zeta^2(z) = \sum_{h \in H} d(h) h z^{\ell(h)}$ with $d(.)$ the number of divisors

- $\frac{\zeta'(z)}{\zeta(z)} = - \sum_{h \in H} \Lambda(h) h z^{\ell(h)}$
I) Generalization of number theory: $\zeta(z) = \det(I - zA(x))^{-1}$

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- $\frac{\zeta'(z)}{\zeta(z)} = -\sum_{h \in \mathcal{H}} \Lambda(h) h z^{\ell(h)}$

- $\log \circ \zeta(z) = -\sum_{h \in \mathcal{H}} \frac{\Lambda(h)}{\ell(h)} h z^{\ell(h)}$
I) Generalization of number theory: $\zeta(z) = \det(I-zA(x))^{-1}$

- $\zeta^2(z) = \sum_{h \in \mathcal{H}} d(h)h z^{\ell(h)}$ with $d(.)$ the number of divisors

- $\frac{\zeta'(z)}{\zeta(z)} = -\sum_{h \in \mathcal{H}} \Lambda(h)h z^{\ell(h)}$

- $\log \circ \zeta(z) = -\sum_{h \in \mathcal{H}} \frac{\Lambda(h)}{\ell(h)}h z^{\ell(h)}$

- ... 

These hold for any graph, for $\mathbb{N}$ in particular
II) Understanding oriented cycle covers

- Covering hike:

$$h = x_{15}x_{35}x_{21}x_{54}x_{51}x_{63}x_{13}x_{23}x_{42}x_{12}x_{26}x_{32}x_{31}x_{45}x_{64}x_{53}x_{24}x_{36}x_{62}x_{46}$$
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{35} x_{21} x_{54} x_{51} x_{63} x_{13} x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{21} x_{54} x_{51} x_{63} x_{13} x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]

```
1  2  3  4  5  6
(15) (35)
```
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{54} x_{51} x_{63} x_{13} x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

▶ Covering hike:

\[ h = x_{51} x_{63} x_{13} x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{63} x_{13} x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]

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II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{13} x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{23} x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{42} x_{12} x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{12}x_{26}x_{32}x_{31}x_{45}x_{64}x_{53}x_{24}x_{36}x_{62}x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{26} x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = x_{32} x_{31} x_{45} x_{64} x_{53} x_{24} x_{36} x_{62} x_{46} \]
II) Understanding oriented cycle covers

- Covering hike:

\[ h = \]

![Graph with nodes and edges]

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II) Understanding oriented cycle covers

- Covering hike:

\[ h = \]

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II) Understanding oriented cycle covers

- Covering hike:

\[ h = \]

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
(12) & (24) & (36) & (46) & (53) & (62) \\
(13) & (23) & (31) & (45) & (51) & (64) \\
(15) & (21) & (35) & (42) & (54) & (63) \\
\end{array}
\]
III) Graph representation of monoids?

- If finitely generated and unique prime decomposition?
- If countable number of primes? (locally finite graphs)
- Minimal representation?
- Extensions to groups? Rings?
OUTLINE

Walks on a graph and linear algebra

Hikes incidence algebra

Spectral analysis

Open hikes
\[
\det(I - zA(x)) = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)
\]
\[
\det(I - zA(x)) = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)
\]
\[
= 1 - \sum_{\lambda \in \text{Sp}(A(x))} \lambda \ z + \sum_{\lambda_1, \lambda_2 \in \text{Sp}(A(x))} \lambda_1 \lambda_2 \ z^2 - ... 
\]
\[ \det(I - zA(x)) = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda) \]

\[ = 1 - \sum_{\lambda \in \text{Sp}(A(x))} \lambda z + \sum_{\lambda_1, \lambda_2 \in \text{Sp}(A(x))} \lambda_1 \lambda_2 z^2 - \ldots \]

\[ = \sum_{k=0}^{n} \sum_{\lambda_1, \ldots, \lambda_k \in \text{Sp}(A(x))} (-1)^k \lambda_1 \ldots \lambda_k z^k \]
\[
\det(I - zA(x)) = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)
\]

\[
= 1 - \sum_{\lambda \in \text{Sp}(A(x))} \lambda z + \sum_{\lambda_1, \lambda_2 \in \text{Sp}(A(x))} \lambda_1 \lambda_2 z^2 - ... 
\]

\[
= \sum_{k=0}^{n} \sum_{\lambda, ..., \lambda_k \in \text{Sp}(A(x))} (-1)^k \lambda_1 ... \lambda_k z^k 
\]

Let \( S = \{ \lambda^{\alpha_1} ... \lambda^{\alpha_k} : \lambda_i \in \text{Sp}(A(x)) \} \)

\[
\det(I - zA(x)) = \sum_{s \in S} \mu_S(s) s^{\ell(s)} 
\]

where \( \mu_S \) is the “all-commutative” Mobius function
Lemma
For all $k = 0, 1, ..., n$

$$\sum_{h \in \mathcal{H}, \ell(h) = k} \mu(h)h = \sum_{\lambda_1, ..., \lambda_k \in \Lambda} (-1)^k \lambda_1 ... \lambda_k = \sum_{s \in S, \ell(s) = k} \mu_S(s)s$$

Equality of the two Mobius functions over equal degree elements
**Lemma**

For all $k = 0, 1, \ldots, n$

$$\sum_{h \in \mathcal{H}, \ell(h) = k} \mu(h) h = \sum_{\lambda_1, \ldots, \lambda_k \in \Lambda} (-1)^k \lambda_1 \ldots \lambda_k = \sum_{s \in S, \ell(s) = k} \mu_S(s) s$$

Equality of the two Mobius functions over equal degree elements

**Theorem (Spectral Mobius inversion)**

For all $k = 0, 1, \ldots, n$

$$\sum_{h \in \mathcal{H}, \ell(h) = k} h = \sum_{s \in S, \ell(s) = k} s$$

Equality of the formal series zeta functions over $\mathcal{H}$ and $S$
Clear connections between hikes and spectrum

- Hike interpretation of co-spectral graphs
- Bounds on spectrum using the hike poset
- Hike structure of special graphs (bipartite, stars, wheels etc...)
- Self-avoiding walks on a lattice
Spectral approach

1. $\zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1}$
**Spectral approach**

1. \( \zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1} \)

2. \( \log \circ \zeta(z) = - \sum_{\lambda \in \text{Sp}(A(x))} \log(1 - z\lambda) \)
SPECTRAL APPROACH

1. $\zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1}$

2. $\log \circ \zeta(z) = -\sum_{\lambda \in \text{Sp}(A(x))} \log(1 - z\lambda)$

$$= -\sum_{\lambda \in \text{Sp}(A(x))} \sum_{k \geq 1} \frac{\lambda^k z^k}{k}$$
**Spectral approach**

1. \( \zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1} \)

2. \( \log \circ \zeta(z) = - \sum_{\lambda \in \text{Sp}(A(x))} \log(1 - z\lambda) \)

\[
= - \sum_{\lambda \in \text{Sp}(A(x))} \sum_{k \geq 1} \frac{\lambda^k z^k}{k} \\
= - \sum_{k \geq 1} \frac{z^k}{k} \sum_{\lambda \in \text{Sp}(A(x))} \lambda^k
\]
Spectral approach

1. \( \zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1} \)

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   \[ = - \sum_{\lambda \in \text{Sp}(A(x))} \sum_{k \geq 1} \frac{\lambda^k z^k}{k} \]
   \[ = - \sum_{k \geq 1} \frac{z^k}{k} \sum_{\lambda \in \text{Sp}(A(x))} \lambda^k \]
   \[ = - \sum_{k \geq 1} \frac{z^k}{k} \text{tr}(A(x)^k) \]
SPECTRAL APPROACH

1. \( \zeta(z) = \text{det}(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1} \)

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   \[= - \sum_{k \geq 1} \frac{z^k}{k} \sum_{h \in \mathcal{H}} \Lambda(h) h \]
1. \( \zeta(z) = \det(I - zA(x))^{-1} = \prod_{\lambda \in \text{Sp}(A(x))} (1 - z\lambda)^{-1} \)

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\]

\[
= -\sum_{k \geq 1} \frac{z^k}{k} \sum_{\lambda \in \text{Sp}(A(x))} \Lambda(h) h = -\sum_{h \in \mathcal{H}} \frac{\Lambda(h)}{\ell(h)} h z^{\ell(h)}
\]
OUTLINE

WALKS ON A GRAPH AND LINEAR ALGEBRA

HIKES INCIDENCE ALGEBRA

SPECTRAL ANALYSIS

OPEN HIKES
**Definition**

For \( i \neq j \), \( h \) is an *open hike* from \( v_i \) to \( v_j \) if \( h \times x_{ji} \) is a closed hike.

- An open hike is an open walk without the connectedness condition.

**Notation**

- \( \mathcal{W}_{ij} \) the set of walks (connected hikes) from \( i \) to \( j \)
- \( S_{ij} \) the set of self-avoiding hikes from \( i \) to \( j \)
PROPOSITION

\[(A(x)^k)_{ij} = \sum_{h \in W_{ij}} h\]
**Proposition**

\[(A(x)^k)_{ij} = \sum_{h \in W_{ij} \atop \ell(h) = k} h\]

**Example**

\[(A(x)^3)_{14} = \]

Diagram: A network of nodes labeled 1 to 7 with directed edges connecting them.
PROPOSITION

\[(A(x)^k)_{ij} = \sum_{h \in \mathcal{W}_{ij}, \ell(h) = k} h\]

EXAMPLE

\[(A(x)^3)_{14} = x_{12}x_{23}x_{34}\]
**PROPOSITION**

\[
(A(x)^k)_{ij} = \sum_{\substack{h \in W_{ij} \\ \ell(h) = k}} h
\]

**EXAMPLE**

\[
(A(x)^3)_{14} = x_{12}x_{23}x_{34} + x_{13}x_{34}x_{44}
\]
THEOREM

\[
\sum_{\substack{d|h \\ d \in \mathcal{H}}} \mu(d) \mathbb{1}\left\{ \frac{h}{d} \in \mathcal{W}_{ij} \right\} = (-1)^{\Omega(h)} \mathbb{1}\left\{ h \in S_{ij} \right\}
\]

- Mobius relation: \( \mu \ast \mathbb{1}\{ . \in \mathcal{W}_{ij} \} = (-1)^{\Omega(.)} \mathbb{1}\{ . \in S_{ij} \} \)

- Walks are just connected hikes
PROOFS

1.

\[
\text{adj}(I - zA(x)) \cdot (I - zA(x)) = \det(I - zA(x)) \cdot I
\]
Proofs

1. \[ \text{adj}(I - zA(x)) \cdot (I - zA(x)) = \det(I - zA(x)) \cdot I \]

2. \[ \text{adj}(I - zA(x)) = \det(I - zA(x)) \cdot (I + zA(x) + z^2A(x)^2 + ...) \]
**Proofs**

1. $$\text{adj}(l - zA(x)) . (l - zA(x)) = \text{det}(l - zA(x)) . l$$

2. $$\text{adj}(l - zA(x)) = \text{det}(l - zA(x)) . (l + zA(x) + z^2A(x)^2 + ...)$$

3. $$\sum_{h: i \to j} (-1)^{\Omega(h)} h z^{\ell(h)} = \left( \sum_{h \in \mathcal{H}} \mu(h) h z^{\ell(h)} \right) \left( \sum_{w: i \to j} w z^{\ell(w)} \right)$$

   *h self-avoiding*
\section*{Proofs}

1. \[ \text{adj}(I - zA(x)) \cdot (I - zA(x)) = \det(I - zA(x)) \cdot I \]

2. \[ \text{adj}(I - zA(x)) = \det(I - zA(x)) \cdot (I + zA(x) + z^2A(x)^2 + ...) \]

3. \[ \sum_{h : i \rightarrow j} (-1)^{\Omega(h)} h z^{\ell(h)} = \left( \sum_{h \in \mathcal{H}} \mu(h) h z^{\ell(h)} \right) \left( \sum_{w : i \rightarrow j} w z^{\ell(w)} \right) \] for \( h \) self-avoiding

4. \[ \sum_{h \in \mathcal{H}} (-1)^{\Omega(h)} \mathbb{1}_{\{h \in S_{ij}\}} h z^{\ell(h)} = \sum_{h \in \mathcal{H}} \mu \ast \mathbb{1}_{\{\cdot \in \mathcal{W}_{ij}\}}(h) h z^{\ell(h)} \]
GRACIAS POR SU ATENCION