Harmonic Oscillator Driven by a Quantum Current

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We show that under certain circumstances a simple quantum harmonic oscillator driven by a quantum current evolves to unique pure states even if started as a mixed state. In various limits, these states exhibit nonclassical properties such as sub-Poissonian statistics, or more interestingly resemble macroscopic superpositions.

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The simple harmonic oscillator occupies a privileged position both in classical and in quantum physics. It is important in describing small oscillations about equilibrium positions and hence gives a description of many wave phenomena. In particular, because of its one-to-one correspondence with a single mode of the electromagnetic field, it is the central paradigm of QED and quantum optics.

In recent years, it has become possible to almost perfectly isolate single quantum harmonic oscillators (HO) from their environment, and in particular from dissipation: The use of single-mode superconducting cavities enables detailed investigations of the dynamics of this simplest of all quantum systems.[1,2] Such studies are bound to yield a deeper understanding of quantum mechanics by pushing it towards the realm of single, isolated systems rather than ensembles.

It is well known that a HO at zero temperature driven by a classical current and subject to dissipation evolves towards a unique pure state, a coherent state.[3] Dissipation ensures that the initial character of the HO's state is transient and becomes replaced by that state which balances the gain from the classical current with the losses. With hindsight, the fact that the photon statistics of single-mode lasers far above threshold are nearly Poissonian can be traced back to this property.

In this Letter we consider the evolution of a HO coupled to a *quantum* current. We show that under appropriate conditions it is driven towards a new class of pure states. In various limits these states reduce to coherent states or to number states. More interestingly, they can also acquire the character of macroscopic superpositions. The evolution to a pure state also occurs for mixed initial states of the HO.

The current that we consider consists of a string of spin-$\frac{1}{2}$ particles, each interacting with the oscillator for a time τ. This is an idealized description of the excitation of the single-mode superconducting cavities used, e.g., in micromasers.[1,2] There, the cavity mode interacts with a low-density stream of two-level atoms such that only one atom at a time is present inside the resonator.

After M spins, the reduced density matrix for the oscillator alone is given by

$$\rho(M\tau)=\mathrm{Tr}_s[U(\tau)\rho_s\cdots\mathrm{Tr}_s[U(\tau)\rho_s\mathrm{Tr}_s[U(\tau)\rho_s\rho(0)U^\dagger(\tau)]U^\dagger(\tau)]\cdots U^\dagger(\tau)]\,, \quad (1)$$

where $U(\tau)=\exp(-iH\tau/\hbar)$ is the evolution operator for the spin-oscillator system, ρ_s is the density matrix of the subsequent spin-$\frac{1}{2}$ particles at the beginning of their interaction with the HO, Tr_s is a partial trace over the spin variables, and H is the Jaynes-Cummings Hamiltonian[4]

$$H=\hbar\omega a^\dagger a+\hbar\omega\sigma_z+\hbar\kappa(a^\dagger\sigma_-+a\sigma_+)\,. \quad (2)$$

Here, $a,a^\dagger$ are the oscillator annihilation and creation operators, σ_z, σ_-, and σ_+ are Pauli matrices, and $\hbar\kappa$ is the oscillator-spin coupling constant. We assume resonance between the oscillator and spin frequencies. The eigenenergies and eigenstates of H are known.[4] In writing Eq. (1) it is assumed that the spin-$\frac{1}{2}$ density matrix ρ_s is identical for all spins as they start their interaction with the HO. This implies that if ρ_s describes a coherent superposition of spin states, the phase of the superposition must be the same for all spins. The M partial traces in Eq. (1) express the fact that successive spins interact with the HO for a time τ only. Such traces are sometimes referred to as nonselective measurements:[5] We know that the spins stay in the cavity for a time τ only, but we do not measure the state in which they exit the resonator.

While numerically solving Eq. (2) for

$$\rho_s=(\alpha|a\rangle+\beta|b\rangle)(\alpha^*\langle a|+\beta^*\langle b|)\,, \quad (3)$$

with $|\alpha|^2+|\beta|^2=1$, $|a\rangle$ and $|b\rangle$ being the upper and lower spin states, we found that the reduced density matrix for the oscillator alone evolves towards a pure (zero-entropy) steady state if the spin-field interaction time τ satisfies the "trapping" condition[6] $\kappa(n_1+l)^{1/2}\tau=\pi$ (n_1

integer) and the initial field excitation is limited to Fock states $|m\rangle$ with $m \le n_1$. An example of the final state reached by the HO under such conditions is given in Fig. 1, together with the dynamics of its entropy S.

The evolution of the HO density matrix towards a pure state is a surprising result indeed: In general, partial traces such as appear in Eq. (1) are expected to lead to mixed rather than pure states. What happens instead in the present case is that the harmonic oscillator appears to benefit from a transfer of coherence from the spins.

Under the Jaynes-Cummings dynamics, the evolution of an arbitrary state of the combined oscillator–two-level atom system is given by

$$\sum_n s_n |n\rangle(\alpha|a\rangle+\beta|b\rangle) \rightarrow \sum_n s_n\{\alpha\cos[\kappa(n+1)^{1/2}\tau]|n\rangle + i\beta\sin(\kappa\sqrt{n}\tau)|n-1\rangle\}|a\rangle + \sum_n s_n\{\beta\cos(\kappa\sqrt{n}\tau)|n\rangle + i\alpha\sin[\kappa(n+1)^{1/2}\tau]|n+1\rangle\}|b\rangle . \quad (4)$$

Trapping states[6] of the oscillator play an essential role in its dynamics. They are immediately apparent from Eq. (4): If for some $n=n_q$ we have

$$\kappa\sqrt{n_q}\tau = q\pi, \quad q \text{ integer}, \quad (5a)$$

then the downward coupling between $|n_q\rangle$ and $|n_q-1\rangle$ vanishes and the phase-space regions below and above $|n_q\rangle$ are dynamically disconnected. We call $|n_q\rangle$ a downward $q\pi$-trapping state. Similarly, a state such that

$$\kappa(n_q+1)^{1/2}\tau = q\pi, \quad q \text{ integer} \quad (5b)$$

is an upward $q\pi$-trapping state. Equations (5) show that the state immediately following an upward $q\pi$-trapping state is always a downward $q\pi$-trapping state. An important property of trapping states is that since they separate the phase space of the HO into disconnected blocks, initial conditions within one block cannot leak into others. This is the essential ingredient in obtaining normalizable steady-state states of the HO for arbitrary spin initial conditions.

Since the driven oscillator's dynamics can be handled separately in disconnected phase-space blocks we concentrate on initial conditions within one block only. Under more general initial conditions the HO always evolves towards a mixed state, since the dynamics prohibits the buildup of coherences between disconnected blocks.

We proceed by noting that if the HO evolves towards a pure state, its properties can be determined by a self-consistency argument. We assume that the HO is in the pure state

$$|f\rangle = \sum_n s_n |n\rangle \quad (6)$$

after interaction with a given spin, and require that it remains in this same state (within an overall phase) after interaction with the next spin. This requires that the state of the composite system at time τ factorizes into a tensor product of $|f\rangle$ times a pure state of the two-level atom:

$$|f\rangle(\alpha|a\rangle+\beta|b\rangle) \rightarrow e^{i\phi}|f\rangle(\alpha'|a\rangle+\beta'|b\rangle) . \quad (7)$$

Here ϕ is an overall phase and α',β', with $|\alpha'|^2+|\beta'|^2=1$, are coefficients to be determined. Combining Eqs. (6) and (7) yields two equations that must be satisfied for all n's. If the state of the HO is initially confined between a downward $2q\pi$-trapping state and a subsequent

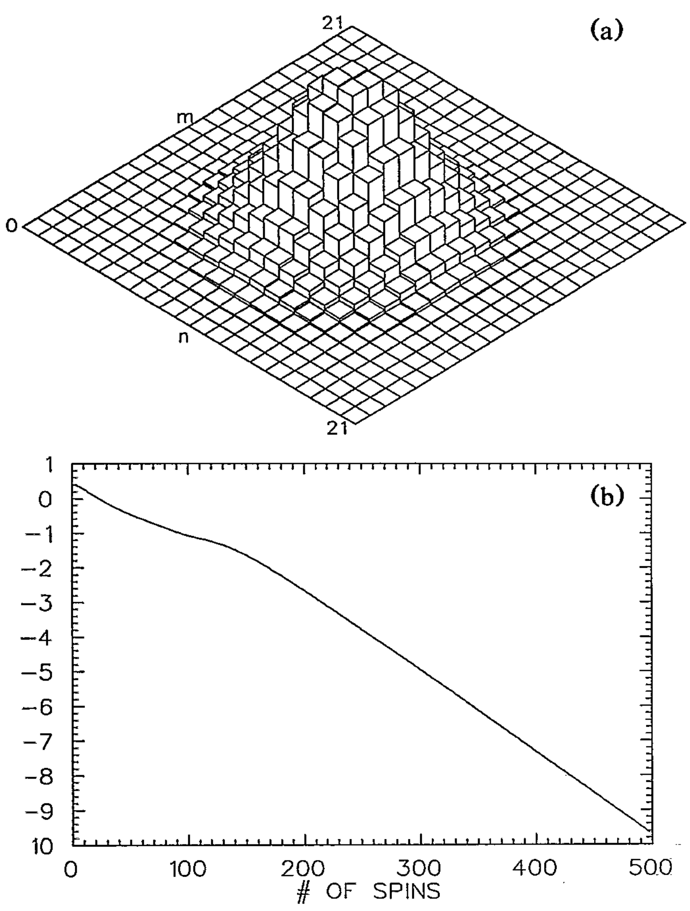

FIG. 1. (a) Modulii of the steady-state density matrix elements $\rho_{nm} = \langle n|\rho|m\rangle$ of the pure state reached by the harmonic oscillator, for $|\alpha|^2 = 0.8$, $n_1 = 21$, and $\kappa\tau = \pi/\sqrt{22}$. The system was started from a thermal mixed state with a mean excitation $\langle n\rangle = 3$. The initial distribution has been slightly truncated and renormalized to avoid having any population above the level $n=21$. (b) Entropy of the HO (on a logarithmic scale) as a function of the number of spins having interacted with it.

upward $(2p+1)\pi$-trapping state (q,p integer) this leads to the unique solution characterized by

$$e^{i\phi}=\pm 1,\quad \alpha'=\mp\alpha,\quad \beta'=\pm\beta. \tag{8a}$$

For the case where the initial state is confined between a downward $(2q+1)\pi$-trapping state and a subsequent upward $2p\pi$-trapping state, we find

$$e^{i\phi}=\pm 1,\quad \alpha'=\pm\alpha,\quad \beta'=\mp\beta. \tag{8b}$$

No other zero-entropy steady states are possible. Choosing the overall phasor $\exp(i\phi)$ to be unity, Eq. (8a) can be interpreted as a nutation of the upper-state probability amplitude by π and Eq. (8b) as a nutation of the lower-state probability amplitude by π. Under these conditions the successive spins exit the cavity with precisely the same energy as they entered it with. Note, however, that there is more than just conservation of energy involved in reaching a steady state of the HO. This is seen readily by noting that $|\alpha'|^2=|\alpha|^2$, $|\beta'|^2=|\beta|^2$ alone is not sufficient to satisfy condition (7). Energy conservation involves only the first moment of the state, while Eq. (7) involves all moments.

Equations (8) lead to simple recurrence relations for the probability amplitudes s_n of Eq. (6). We find readily

$$s_n=-i\frac{e^{i\phi}-(\alpha/\alpha')\cos(\kappa\sqrt{n}\,\tau)}{(\beta/\alpha')\sin(\kappa\sqrt{n}\,\tau)}s_{n-1}, \tag{9}$$

with the corresponding photon statistics

$$|s_n|^2=|\alpha/\beta|^2\cot^2(\kappa\sqrt{n}\,\tau/2)\,|s_{n-1}|^2 \tag{10a}$$

and

$$|s_n|^2=|\alpha/\beta|^2\tan^2(\kappa\sqrt{n}\,\tau/2)\,|s_{n-1}|^2 \tag{10b}$$

for cases (8a) and (8b), respectively. For lack of a better name, we call these states cotangent and tangent states of the HO. Note that the cotangent states are clearly of most practical interest, since physically relevant initial conditions typically include the vacuum state $|0\rangle$ (which is a downward 0π-trapping state).

The detailed properties of the tangent and cotangent states of the HO will be presented in a future publication. Here, we limit our discussion to some of their most striking features. Consider first such states bound between $|0\rangle$ and a π-trapping state $|n_1\rangle$. For fully inverted spins ($\alpha=1$) the system evolves precisely towards the Fock state $|n_1\rangle$.[6] For $\alpha=0$, in contrast, it asymptotically reaches the vacuum state $|0\rangle$. For intermediate situations, the cotangent state is sub-Poissonian, as illustrated in Fig. 2.

For interaction times short enough that $\kappa\tau\ll 1/\sqrt{n}$ for all relevant number states in Eq. (10a), the cotangent state reduces to a coherent state with Poisson photon

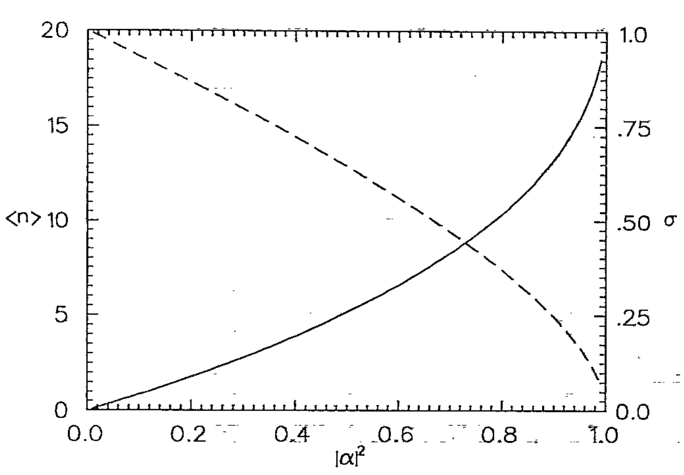

FIG. 2. Solid line: Mean excitation $\langle n\rangle$ of the cotangent state (10a) as a function of the spin excitation $|\alpha|^2$. Dashed line: Normalized second moment $\sigma=(\langle n^2\rangle-\langle n\rangle^2)/\langle n\rangle$, showing sub-Poissonian statistics when $\sigma<1$. Here, $n_1=21$ and $\kappa\tau=\pi/\sqrt{22}$.

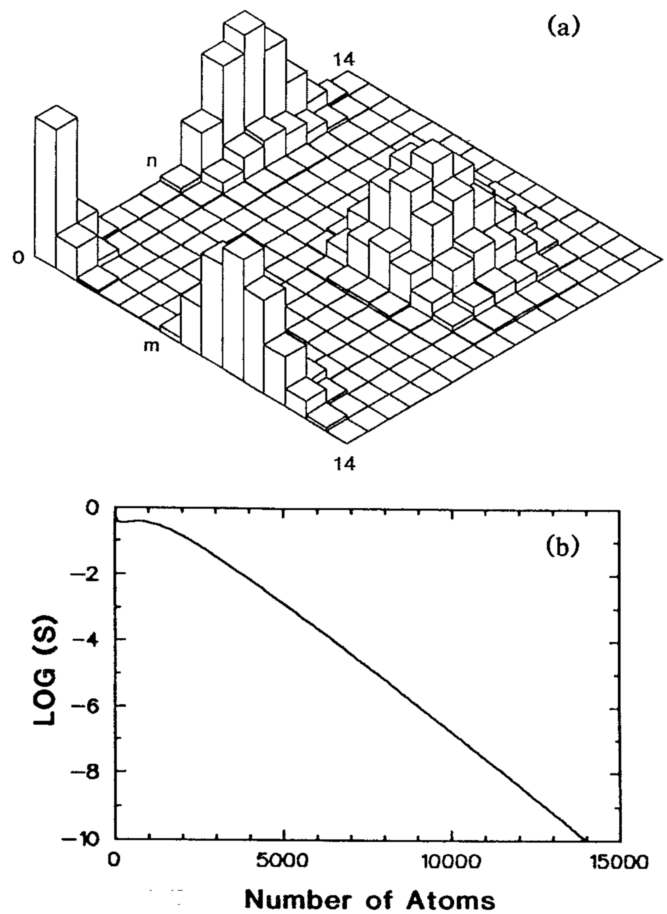

FIG. 3. (a) Modulii of the steady-state density matrix elements $\rho_{nm}=\langle n|\rho|m\rangle$ of the pure state reached by the harmonic oscillator, for $|\alpha|^2=0.3$, $n_2=14$, and $\kappa\tau=3\pi/\sqrt{15}$. The system was started from a thermal mixed state, truncated beyond $n=14$, with a mean excitation $\langle n\rangle=1$. (b) Evolution of the HO entropy (logarithmic scale) as a function of the number of spins.

statistics

$$|s_n|^2 = |2\alpha/\beta\kappa\tau|^2 n^{-1} |s_{n-1}|^2, \quad (11)$$

and mean photon number $\langle n\rangle = |2\alpha/\beta\kappa\tau|^2$. In this limit the HO acts as if driven by a classical current. Note, however, that the short-time condition $\kappa\tau \ll 1/\sqrt{n}$, which guarantees that no significant correlations develop between the successive spins and the HO, is a statement of the semiclassical approximation[7] valid for arbitrary spin initial conditions. This is a less stringent condition than the classical limit known to occur when spin-$\frac{1}{2}$ particles remain close to the ground state, $|\alpha| \ll 1$.[7] It is also important to realize that for arbitrary spin and HO initial conditions, and in particular for inverted spins $(\alpha > \beta)$, the oscillator does feel the presence of the trapping state n_1 during its approach to steady state. This evolution, which obviously takes the HO through number states such that the condition $\kappa\tau \ll 1/\sqrt{n}$ is not fulfilled, is a clear indication of the importance of the "granular nature" of the HO and of quantum dynamics at play. The steady-state limit and the limit of small interaction times *do not* usually commute.

An enormous wealth of states with novel properties can be generated when the phase-space evolution is bounded by higher trapping states. For instance, cotangent states confined between the vacuum and 3π-trapping states acquire properties reminiscent of "macroscopic superpositions," as illustrated in Fig. 3. The final pure state shown in Fig. 3 evolved dynamically from a thermal mixed state and the inset shows the evolution of the HO entropy as a function of the number of spins having interacted with it. This illustrates that macroscopic superpositions can indeed be generated under the system dynamics starting from a mixed state. Such states are of considerable current interest in investigations on the foundations of quantum mechanics and measurement theory.

In conclusion, we have shown that a HO driven by a stream of spin-$\frac{1}{2}$ particles can evolve towards pure states even for mixed initial states. The existence of these new states relies explicitly on the discrete nature of the quantum states of the harmonic oscillator. In various limits, they exhibit nonclassical properties such as sub-Poissonian statistics, or more interestingly resemble macroscopic superpositions. As such they may provide new testing grounds for fundamental tests of quantum mechanics and measurement theory.

We recognize that the quantum current considered here calls for a precisely phased stream of spins, a formidable experimental task indeed. However, there is no fundamental limit prohibiting the construction of such a current. We hope that the wealth of novel quantum states that can be studied in such systems will provide motivation enough to attempt this experimental tour de force.

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