Automorphisms of transition graphs for elementary cellular automata

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The *transition graph* of a cellular automaton (CA) is a graphical representation of the CA's global dynamics. Studying automorphisms of transition graphs allows us to identify symmetries in this global dynamics. We conduct a computational study of numbers of automorphisms for the elementary cellular automata (ECAs) on finite lattices. The ECAs are partitioned into three classes, depending on how the number of automorphisms varies as a function of the number of cells in the lattice. While one of these classes contains the majority of the ECAs, and encompasses dynamical behaviour ranging from trivial to complex, the other two classes identify those ECAs whose local rule is a nontrivial linear function of its inputs, as well as those ECAs capable of producing chaotic dynamics from ordered initial configurations, and a small number of other "exceptional" ECAs.

Key words: Elementary cellular automata, classification, transition graphs, automorphisms, symmetry

1 INTRODUCTION

An *elementary cellular automaton (ECA)* is a 1-dimensional cellular automaton, whose state set is $\mathbb{Z}_2 = \{0, 1\}$ and where the neighbourhood of a cell consists of the cell itself and those cells immediately adjacent to the left and right. The ECAs were introduced by Stephen Wolfram in 1983 [5], and subsequently studied extensively by Wolfram and others [8, 9]. More recent

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interest in the ECAs has been fuelled by Cook's proof that at least one of them (*rule 110* according to the numbering scheme described in Section 2) is Turing complete [1].

Wolfram [6] proposed that the ECAs (and CAs in general [4]) can be divided into four classes, depending on whether the long-term dynamics starting from a random initial configuration is homogeneous, periodic, chaotic or complex. This "classification" relies on visual inspection of the CA's evolution.

In CAs, as in many complex systems, complex "global" dynamics can emerge from simple "local" behaviour. Attempts to predict the complexity of the global dynamics based solely on properties of the local update rule are often unsuccessful, at least for ECAs. For example, Langton [3] points out that, although his λ parameter is a good predictor of complexity for CAs with more states and larger neighbourhoods, it is *"only roughly correlated with dynamical behavior"* for ECAs. Thus we might expect that a more global approach is required; indeed, measures such as Wuensche's Z parameter [10], which arises from an algorithm for finding the pre-images of a given configuration, seem to be improvements over Langton's λ parameter.

We propose a partial classification of the ECAs based on a property of their global dynamics. A *transition graph* is a representation of a CA's global dynamics as a directed graph. Transition graphs tend to have a high degree of symmetry (or more formally, admit a large number of *automorphisms*). We can compute the number of automorphisms for a given transition graph; our classification is based on studying how the number of automorphisms varies with the number of cells on which the ECA operates.

2 ELEMENTARY CELLULAR AUTOMATA

An ECA has state set $\mathbb{Z}_2 = \{0, 1\}$, and each cell has three neighbours including itself. Thus the *local update rule* for an ECA is a function $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2$. The 256 possibilities for f are assigned to the numbers 0 to 255 by interpreting the string

$$f(1,1,1)\ f(1,1,0)\ \ldots\ f(0,0,0) \tag{1}$$

as a number in binary notation and converting to decimal.

A *configuration* of a CA is an assignment of states to cells. In this paper, we focus on ECAs on the finite lattice of N cells with periodic boundary condition; a configuration is thus an element of the set $\mathbb{Z}_2^N$, which is effectively the set of all functions from $\mathbb{Z}_N$ to $\mathbb{Z}_2$. The local update rule f extends to a *global map* $F : \mathbb{Z}_2^N \to \mathbb{Z}_2^N$ in the usual way.

We consider two rules to be *equivalent* if one can be obtained from the other by left-right reflection, by exchanging states 0 and 1, or by performing those two transformations sequentially. This partitions the space of 256 rules into 88 equivalence classes. Rules in the same equivalence class exhibit equivalent dynamics (in particular, their transition graphs are isomorphic), thus we obtain the set of 88 “essentially different” ECA rules by choosing an element from each equivalence class.

3 TRANSITION GRAPHS

The *transition graph* for an ECA is the digraph $(\mathbb{Z}_2^N, \mathcal{E})$ whose edge set is

$$\mathcal{E} = \left\{(s, F(s)) \ : \ s \in \mathbb{Z}_2^N\right\}. \tag{2}$$

In other words, the transition graph is the *functional graph* [2] for the global map F: there is an edge from vertex x to vertex y if and only if $y = F(x)$. It is clear that every vertex in the transition graph has out-degree 1. This forces the transition graph to have a “circles of trees” topology: the graph consists of one or more disjoint cycles, with a (possibly single-vertex) tree rooted at each vertex in each cycle. An example of a transition graph is shown in Figure 1.

In terms of dynamical systems, the vertices of the transition graph form the *phase space* of the CA, and paths in the graph are trajectories in this phase space. Each “circle” (a cycle and its trees) is a *basin of attraction*, and thus we call the disjoint components of the transition graph *basins*.

A high degree of symmetry is apparent in the transition graphs for the ECAs. Many of the individual basins are isomorphic to each other, many basins within themselves exhibit rotational symmetry, and trees tend to contain subtrees which are isomorphic to each other. These symmetries are all examples of *automorphisms*: isomorphisms of the graph onto itself. It turns out that automorphisms of transition graphs are, in a sense, “symmetries” of the global dynamics of the corresponding ECAs.

Definition 1. Consider $F, G : X \to X$, where X is a discrete topological space. A bijection $\alpha : X \to X$ is a *topological conjugation* from F to G if $G \circ \alpha = \alpha \circ F$.

Theorem 1. *Consider a function $F : X \to X$. A bijection $\alpha : X \to X$ is an automorphism of the functional graph of F if and only if α is a topological conjugation from F to F.*

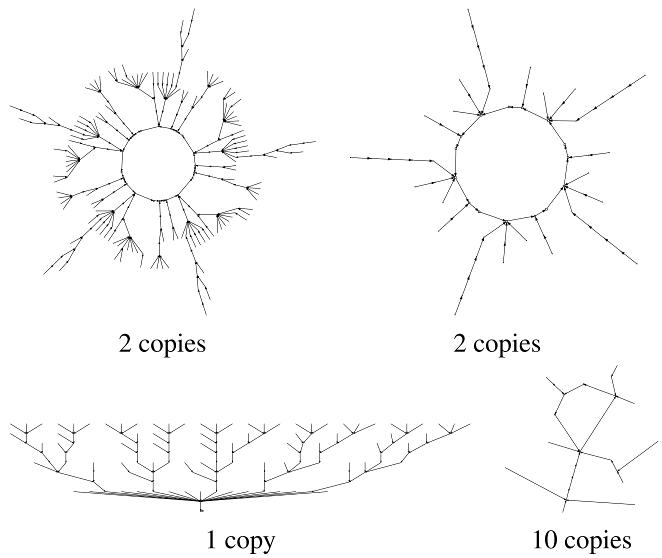

FIGURE 1
Transition graph for ECA 110 on the periodic lattice $\mathbb{Z}_{10}$.

This follows immediately from the definitions. Applying this theorem to transition graphs, we obtain the following result:

Corollary 1. *For a given ECA and a given lattice size N, a bijection $\alpha : \mathbb{Z}_2^N \to \mathbb{Z}_2^N$ is an automorphism of the transition graph if and only if α is a topological conjugation from F to F, where F is the ECA's global map.*

In other words, α is an automorphism if and only if $F \circ \alpha = \alpha \circ F$. In a sense, α is a symmetry of the CA: consider the evolution of the ECA from a given initial configuration s, and the evolution from the configuration $\alpha(s)$. The latter evolution is simply the image under α of the former. Compare this with the concept of symmetry in physical laws, where the only effect of applying a symmetry transformation to an initial condition is to apply that same transformation to the subsequent dynamics of the system.

There is one class of transformations which must be automorphisms for all ECAs. For any integer k, we define the *k-shift transform* $\sigma_k : \mathbb{Z}_2^N \to \mathbb{Z}_2^N$

by

$$\sigma_k(x_0 \dots x_{N-1}) = x_{0+k} \dots x_{N-1+k} \quad \text{for all } x_0 \dots x_{N-1} \in \mathbb{Z}_2^N, \tag{3}$$

where the arithmetic in the subscripts takes place in $\mathbb{Z}_N$; that is, modulo N. So σ_k shifts the entire configuration to the left by k cells, obeying the periodic boundary condition. It is easy to see that σ_k is a bijection, and it follows from the definition of a CA that σ_k must be a topological conjugation from the global map to itself. Thus, by Corollary 1, $\{\sigma_k : k \in \mathbb{Z}_N\}$ must be a subgroup of the group of automorphisms of the transition graph for any ECA.

Similarly, we find that the *reflection transform* ρ and the *conjugation transform* c, defined by

$$\rho(x_0 \dots x_{N-1}) = x_{N-1} \dots x_0 \tag{4}$$

and

$$c(x_0 \dots x_{N-1}) = (1 - x_0) \dots (1 - x_{N-1}), \tag{5}$$

are automorphisms for precisely those ECAs which exhibit "left-right symmetry" and "0-1 symmetry" respectively.

The shift, reflection and conjugation transforms are the most obvious "geometric" symmetries, but by no means are they the only symmetries which an ECA may possess. How many more are there? In the next section, we describe a simple algorithm to count them.

4 COUNTING AUTOMORPHISMS

Let $A(f, N)$ denote the order of the automorphism group for the transition graph of ECA rule f on the lattice of N cells. It is easy to find $A(f, N)$ for some ECAs. For rule 204 (the identity rule), every permutation of configurations is an automorphism, and so there are $2^N!$ automorphisms in total (and clearly this is an upper bound on the number of automorphisms for any ECA). For rule 0, the automorphisms are precisely those permutations which fix the all-zeros configuration, and so there are $(2^N - 1)!$ automorphisms.

The following results describe how $A(f, N)$ may be computed in the general case, by induction over the "circles of trees" structure of the transition graph.

Lemma 1. *Consider a tree with root vertex v, and let $\{u_1, \dots, u_n\}$ be the set v's* children*:*

$$\{u_1, \dots, u_n\} = \left\{u \in \mathbb{Z}_2^N \;:\; (u, v) \in \mathcal{E},\; u \textit{ is not in a cycle}\right\}. \tag{6}$$

Denote by $\{u_i\}/\cong$ the set of isomorphism classes of $\{u_1, \ldots, u_n\}$, considering two vertices to be isomorphic if the subtrees rooted at those vertices are isomorphic. Denote by $A(v)$ the order of the isomorphism group of the tree rooted at v. Then $A(v)$ is defined inductively by

$$A(v) = \left(\prod_{I \in \{u_i\}/\cong} |I|! \right) \left(\prod_{i=1}^{n} A(u_i) \right). \tag{7}$$

Lemma 2. *Consider a basin B in a transition graph, and suppose that the vertices of the basin's cycle are $\langle v_1, \ldots, v_m \rangle$. Let $q > 0$ be minimal such that, for all i, the tree rooted at v_i is isomorphic to the tree rooted at v_{i+q}. Clearly q must divide m. Then the order of the automorphism group of the basin is*

$$A(B) = \frac{m}{q} \prod_{i=1}^{m} A(v_i), \tag{8}$$

with $A(v_i)$ as defined in Lemma 1.

Theorem 2. *Consider a transition graph composed of basins $\{B_1, \ldots, B_k\}$, and denote the set of isomorphism classes of $\{B_1, \ldots, B_k\}$ by $\{B_i\}/\cong$. Then the order of the automorphism group of the transition graph is*

$$A(f, N) = \left(\prod_{I \in \{B_i\}/\cong} |I|! \right) \left(\prod_{i=1}^{k} A(B_i) \right), \tag{9}$$

with $A(B_i)$ as defined in Lemma 2.

Intuitively, an automorphism of the transition graph is composed of:

1. A permutation ϖ of $\{B_1, \ldots, B_k\}$ which preserves isomorphism classes, so that $\varpi(B_i)$ is isomorphic to B_i for all i;
2. For each i, an isomorphism from B_i to $\varpi(B_i)$; or equivalently, an automorphism on B_i.

Thus the total number of automorphisms is given by Equation 9, with the two products on the right-hand side corresponding to the total numbers of choices for items 1 and 2. The argument for Lemma 1 is similar. For Lemma 2, we note that an automorphism on basin B is composed of:

1. A cyclic shift of $\langle v_1, \ldots, v_m \rangle$ which preserves isomorphism classes;
2. For each i, an automorphism on the tree rooted at v_i.

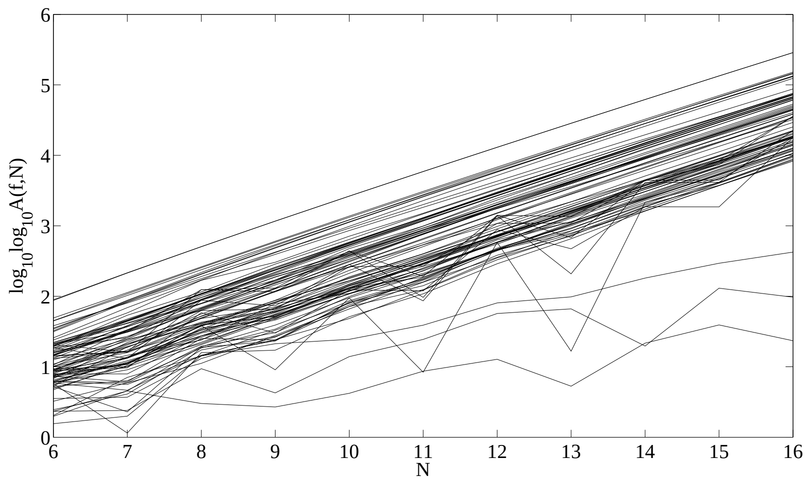

FIGURE 2
Plot of $\log_{10}\log_{10} A(f,N)$ against N, for $6 \leq N \leq 16$ and for all 88 essentially different ECA rules.

A cyclic shift of the type required must be a shift by a multiple of q, and conversely a shift by a multiple of q must preserve isomorphism classes by definition of q. Thus there are m/q such shifts.

Computing $A(f,N)$ requires that the entire transition graph of 2^N vertices be generated and traversed, so the computational time complexity is at least exponential with respect to N. We have computed $A(f,N)$ for the 88 essentially different ECAs for all $N \leq 16$, and for $N = 17$ for selected rules, and this seems to approach the limit of what can be computed on a modern desktop PC within a reasonable length of time. Furthermore, the exponential complexity means that even an orders-of-magnitude increase in computational power would not significantly increase this limit. However, the data we are able to generate are sufficient to make some interesting observations.

5 RESULTS

Figure 2 plots $A(f,N)$ for each of the 88 essentially different ECAs, and Table 1 gives selected numerical values. By inspection of Figure 2, the ECAs can be partitioned into three sets depending on the behaviour of $A(f,N)$ with respect to N: the first and largest set shows an approximate linear relationship

Set	Rule	N 10	11	12	13	14	15	16	17
$\mathcal{L}$	204	3.422	3.770	4.115	4.455	4.792	5.126	5.458	5.788
	15	2.419	2.430	3.034	3.041	3.645	3.649	4.254	4.257
	60	1.976	0.923	2.765	1.221	3.335	3.616	3.994	3.053
$\mathcal{Z}$	90	2.451	1.935	3.141	2.318	3.614	3.617	4.354	4.283
	105	2.644	1.984	3.149	2.833	3.646	3.868	4.555	4.311
	150	2.642	2.282	3.149	3.133	3.646	3.923	4.555	4.612
	154	2.091	2.077	2.945	2.674	3.269	3.268	4.252	3.856
	30	0.624	0.937	1.106	0.724	1.336	1.594	1.369	1.296
$\mathcal{C}$	45	1.144	1.388	1.756	1.822	1.293	2.115	1.987	1.833
	106	1.389	1.590	1.906	1.991	2.258	2.468	2.627	2.849

TABLE 1
Values, to 3 decimal places, of $\log_{10}\log_{10} A(f,N)$ for rule 204 (the identity rule, for which $\log_{10}\log_{10} A(f,N)$ is maximal) and the rules in sets $\mathcal{Z}$ and $\mathcal{C}$.

between $\log_{10}\log_{10} A(f,N)$ and N, the second shows alternation between larger and smaller values for successive values of N, and the third shows a reduced rate of growth with neither linear nor alternating behaviour. We denote these sets $\mathcal{L}$, $\mathcal{Z}$ and $\mathcal{C}$ respectively.*

5.1 Set $\mathcal{L}$: approximate linear relationship

The majority of ECAs show an approximately linear relationship between $\log_{10}\log_{10} A(f,N)$ and N. Furthermore, all of the lines have approximately the same gradient.

Set $\mathcal{L}$ contains examples from all four of Wolfram's classes, including the identity rule 204 and the "Turing complete" rule 110.

5.2 Set $\mathcal{Z}$: alternating between large and small values

Figure 3 (a) plots $A(f,N)$ for ECA rules 15, 60, 90, 105, 150, and 154. For $N \leq 14$, these rules are characterised by $A(f,N)$ alternating between large values for even N, and smaller values for odd N. This pattern seems to break down, or at least become less pronounced, for $15 \leq N \leq 17$.

Definition 2. A local update rule for an ECA is *linear* if it has the form

$$\delta(x,y,z) = ax + by + cz + d \tag{10}$$

* These symbols were chosen because the relationship between $\log_{10}\log_{10} A(f,N)$ and N is either "linear", "zigzag", or "chaotic".

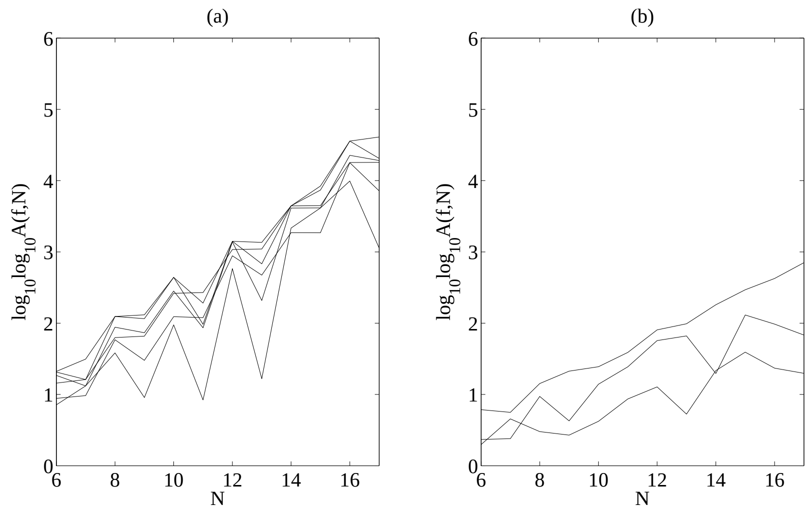

FIGURE 3
As Figure 2, but for $6 \leq N \leq 17$, and only showing the ECAs in (a) set $\mathcal{Z}$; (b) set $\mathcal{C}$.

for some constants $a, b, c, d \in \mathbb{Z}_2$. The rule is *non-trivial* if more than one of the coefficients $\{a, b, c\}$ is nonzero, otherwise it is *trivial*.

Set $\mathcal{Z}$ contains all four of the non-trivial linear ECAs (rules 60, 90, 105 and 150), along with one of the trivial linear ECAs (rule 15). The four remaining trivial linear rules (0, 51, 170 and 204) are in $\mathcal{L}$.

The single non-linear rule in $\mathcal{Z}$, rule 154, shares one property with the non-trivial linear rules: from an initial configuration consisting of a single cell in state 1, a self-similar "Sierpiński gasket" pattern is produced. On other initial configurations, rule 154 produces periodic patterns, in contrast to the chaotic patterns produced by the non-trivial linear rules.

5.3 Set $\mathcal{C}$: neither linear nor alternating

Figure 3 (b) plots $A(f, N)$ for rules 30, 45, and 106. This plot shows neither a linear nor an oscillating relationship, and the rate of growth of $A(f, N)$ with respect to N seems significantly lower than that for the ECAs in sets $\mathcal{L}$ and $\mathcal{Z}$.

Wolfram [7] identifies rules 30 and 45 as being particularly suited to random number generation by ECAs: if the initial configuration assigns state 1 to a single cell and state 0 to all others, then the sequence of states taken

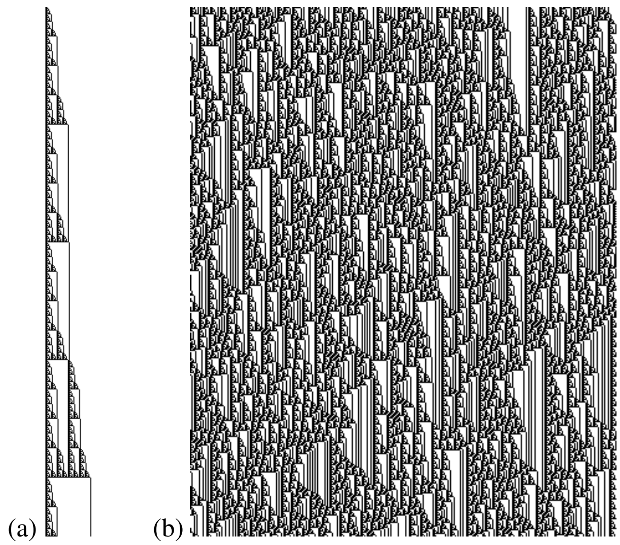

FIGURE 4
Evolution of ECA rule 106 from (a) the configuration $\ldots 001100 \ldots$; (b) a random initial configuration. To make the patterns clearer, each configuration has been offset to the right by one cell relative to the previous configuration.

by that special cell subsequently is, by several measures, a good pseudo-random sequence over $\mathbb{Z}_2$. Rules 30 and 45 do seem to be unique in this respect: other "chaotic" ECAs which fall into Wolfram's class 3 generally only exhibit this property on a random initial configuration; we may think of them as merely "preserving" the randomness already present in the initial configuration, whereas rules 30 and 45 are uniquely capable of "generating" randomness from an ordered initial configuration.

The evolution of rule 106 from a single cell in state 1 is rather uninteresting: the state 1 simply propagates to the left by one cell per generation. However, two adjacent cells in state 1 yield the self-similar pattern shown in Figure 4 (a). On a random initial configuration, rule 106 produces chaotic patterns interspersed with ordered regions, as depicted in Figure 4 (b). Again, rule 106 seems to be unique among the ECAs: self-similar patterns are not uncommon, but the pattern depicted in Figure 4 (a) seems more complex than the "Sierpiński gasket"-like patterns typical of other ECAs.

6 CONCLUSION

By counting automorphisms in transition graphs, and examining how these counts vary with the number of cells N, we have partitioned the ECAs into three classes:

- Set $\mathcal{Z}$, consisting of the "non-trivial" linear rules and a single non-linear rule (154);
- Set $\mathcal{C}$, consisting of the two "chaotic" ECAs and rule 106;
- Set $\mathcal{L}$, consisting of the remainder.

By no means is this a complete classification of the ECAs: the "remainder" set $\mathcal{L}$ contains 79 out of the 88 essentially different ECAs, and seems to span all classes of dynamic behaviour from trivial to complex.

For reasons of computational complexity, we have thus far been unable to study values of N larger than 17. This is barely large enough to support rule 110's "ether" pattern (whose spatial period is 14), let alone the complex particles which travel through this ether to make Cook's construction [1] possible. We conjecture that rule 110, and other "complex" rules, will deviate from the approximately linear trend characterising set $\mathcal{L}$, precisely when N becomes sufficiently large to support the complex structures which make these rules special.

The reader familiar with parameters such as Langton's λ [3] and Wuensche's Z [10] may wonder what the point is of a parameter such as $A(f, N)$ which is so expensive to calculate! We feel that the interesting aspect of this work is not necessarily the numbers themselves, but rather the suggestion of a relationship between "symmetry" and "complexity". The relationship between sets $\mathcal{L}$ and $\mathcal{C}$ already suggest that, for sufficiently large N, chaotic ECAs such as rules 30 and 45 exhibit far fewer symmetries than the more orderly CAs in set $\mathcal{L}$. It is tempting to conjecture that, as N is made larger still, complex ECAs such as rule 110 will exhibit a critical amount of symmetry somewhere in the region between sets $\mathcal{L}$ and $\mathcal{C}$. This would be in keeping with the "edge of chaos" phenomena observed with parameters such as Langton's λ and Wuensche's Z.

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REFERENCES

[1] Matthew Cook. (2004). Universality in elementary cellular automata. *Complex Systems*, 15(1):1–40.

[2] Frank Harary. (1969). *Graph Theory*. Addison-Wesley.

[3] Chris G. Langton. (1990). Computation at the edge of chaos: Phase transitions and emergent computation. *Physica D*, 42:12–37.

[4] Norman H. Packard and Stephen Wolfram. (March 1985). Two-dimensional cellular automata. *Journal of Statistical Physics*, 38(5):901–946.

[5] Stephen Wolfram. (Jul 1983). Statistical mechanics of cellular automata. *Reviews of Modern Physics*, 55(3):601–644.

[6] Stephen Wolfram. (1984). Universality and complexity in cellular automata. *Physica D*, 10:1–35.

[7] Stephen Wolfram. (1986). Random sequence generation by cellular automata. *Advances in Applied Mathematics*, 7:123–169.

[8] Stephen Wolfram, editor. (1986). *Theory and Applications of Cellular Automata*. World Scientific.

[9] Stephen Wolfram. (1994). *Cellular Automata and Complexity: Collected Papers*. Westview Press.

[10] Andrew Wuensche, (1997). Attractor basins of discrete networks. D.Phil thesis, University of Sussex.